Chapter Four

Dynamical Systems

4.1 Introduction

The previous chapter has been devoted to the analysis of systems of first order linear ODE's. In this chapter we will consider systems of first order ODE's that are not linear but are autonomous; i.e., these are systems of the form

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} F_1(x_1,\ldots,x_n) \\ \vdots \\ F_n(x_1,\ldots,x_n) \end{bmatrix} \qquad (1.1)$$

i.e., $\qquad \vec{X}'(t) = \vec{F}\big(\vec{X}(t)\big).$

Recall that a system is said to be autonomous if $\vec{F}\big(\vec{X}(t)\big)$ does not depend explicitly on t. These systems are often referred to as dynamical systems. It will often be the case that the analytic solution for such systems can not be found but a great deal about the solution for the system can be discovered by analyzing the behavior of the solution in the neighborhoods of special points in the domain called critical points. A point $\vec{a}$ in R^n is a **critical point** for this system (1.1) if and only if $\vec{F}(\vec{a}) = \vec{0}$. A critical point is also called a stationary point because if the system is placed at a stationary point, it will remain there since $\vec{X}'(t) = \vec{0}$; i.e., the state is not changing.

A curve $C = \big\{\vec{X}(t) : t \geq 0\big\}$ is called an **orbit** or **trajectory** for the system (1.1) if $\vec{X}(t)$ satisfies $\vec{X}'(t) = \vec{F}\big(\vec{X}(t)\big)$. In the neighborhood of a critical point, the orbits of a nonlinear system may often be approximated by the orbits of an associated linear system. We will show how this associated linear system can be found in the next section. Once the local solutions near each critical point have been approximated, it may then be possible to blend the local pictures to form a global picture of the solution for the nonlinear system.

Suppose $\vec{X}(t)$ is an orbit for a system and $\vec{a} = [a_1,\ldots,a_n]$ is a critical point. Then for any t,

$$\big|\big|\vec{X}(t) - \vec{a}\big|\big| = \sqrt{(x_1(t) - a_1)^2 + \cdots + (x_n(t) - a_n)^2}$$

equals the distance from the point $\vec{X}(t)$ on the orbit, to the critical point $\vec{a}$. We say that the critical point is **stable** if every orbit that originates near $\vec{a}$ remains near $\vec{a}$. More precisely, $\vec{a}$ is stable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\big|\big|\vec{X}(0) - \vec{a}\big|\big| < \delta \qquad \text{implies} \qquad \big|\big|\vec{X}(t) - \vec{a}\big|\big| \leq \varepsilon \ \text{ for all } t \geq 0.$$

We say that the critical point $\vec{a}$ is **asymptotically stable** if every orbit that originates sufficiently near $\vec{a}$ is drawn continually closer to $\vec{a}$. More precisely, $\vec{a}$ is asymptotically stable if for some $\varepsilon > 0$

$$\big|\big|\vec{X}(0) - \vec{a}\big|\big| < \varepsilon \qquad \text{implies} \qquad \big|\big|\vec{X}(t) - \vec{a}\big|\big| \to 0 \ \text{ as } \ t \to \infty.$$

Any critical point that is asymptotically stable is stable but the converse is false. Any critical point that is not stable is said to be **unstable**. At this point, we are going to restrict our considerations to the 2-dimensional case, $n = 2$. This will allow us to gain insight into our

dynamical systems by representing the system orbits graphically.

4.2 Dynamical Systems in two dimensions
In order to discuss nonlinear dynamical systems, it will be helpful to first consider linear dynamical systems. Linear dynamical systems are systems of linear ODE's like those considered in the previous chapter but here we are going to focus more on qualitative information about the solutions rather than trying to construct these solutions.

4.2.1 Linear Dynamical Systems
A 2-dimensional linear system is of the form
$$\vec{U}'(t) = A\vec{U}(t) \qquad (2.1)$$
i.e., $\quad \dfrac{d}{dt}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$

Such a system has one critical point at $\vec{U} = \vec{0}$. We assume that the coefficient matrix A is nonsingular hence $\vec{U} = \vec{0}$ is the only point $\vec{U} = (u, v)$ where the dynamical system is stationary. For a linear dynamical system there are six different types of critical points, which may be characterized according to the nature of the eigenvalues of A. We will list the six types and provide a sketch of the orbits for each of the cases.

● If A has Complex Eigenvalues: $\lambda = \alpha \pm i\beta$

$\left\{\begin{array}{l} (0,0) \text{ is a CENTER } \left(figure\ 1\right) \text{ if } \alpha = 0 \\ (0,0) \text{ is a SPIRAL SINK } \left(figure\ 2a\right) \text{ if } \alpha < 0 \\ (0,0) \text{ is a SPIRAL SOURCE } \left(figure\ 2b\right) \text{ if } \alpha > 0 \end{array}\right\}$

● If A has Real Eigenvalues: $\lambda = \lambda_1, \lambda_2$

$\left\{\begin{array}{l} (0,0) \text{ is a SOURCE } \left(figure\ 3\right) \text{ if } 0 < \lambda_1 < \lambda_2 \\ (0,0) \text{ is a SINK } \left(figure\ 3\right) \text{ if } \lambda_2 < \lambda_1 < 0 \\ (0,0) \text{ is a SADDLE POINT } \left(figure\ 4\right) \text{ if } \lambda_1 < 0 < \lambda_2 \end{array}\right\}$

We will consider some examples to show the structure of the orbits near each of the six types of critical point

1) Center: Let $A = \begin{bmatrix} 0 & 1 \\ -c^2 & 0 \end{bmatrix}$: Then

$$u'(t) = v(t)$$
$$v'(t) = -c^2 u(t)$$

In this case, the eigenvalues of A are imaginary, $\lambda = \pm ic$. Note that the equations of the system imply,
$$c^2 u u'(t) + v v'(t) = c^2 uv - c^2 uv = 0.$$

But

$$c^2 u u'(t) + v v'(t) = \frac{1}{2} \frac{d}{dt}(c^2 u^2 + v^2)$$

hence

$$\frac{d}{dt}(c^2 u^2 + v^2) = 0$$
$$\text{and} \quad c^2 u(t)^2 + v(t)^2 = Const.$$

Evidently, the trajectories or orbits of this system are the curves
$C = \{(u(t), v(t)) : -\infty < t < \infty\}$ in the u-v plane such that $u(t)$ and $v(t)$ solve the
dynamical system, then $c^2 u(t)^2 + v(t)^2 = Const$ implies that the plots of u versus v for this
system are ellipses. The behavior of the system can be understood from studying the plots
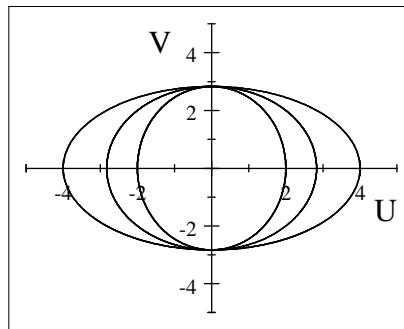of the system orbits in the u-v plane. We refer to these plots as **phase plane portraits** of
the system.

Fig 1 $\mathrm{Re}\,\lambda = 0$

Here we have sketched one orbit for each of the cases, $c^2 = 2, 1, \frac{1}{2}$. Note that since
$u'(t) = v(t)$, u is increasing when v is positive and decreasing when v is negative. This
implies that the orbits are traversed from left to right in the upper half plane and right to left
in the lower half plane. In other words, the curves are traced in the clockwise direction as t
increases. Since the orbits of this system are closed curves, the system assumes the same
values for u and v over and over again. This means that the solutions to the system must be
periodic functions of time. That is, closed curve orbits correspond to periodic solutions for
the dynamical system.

2) Spiral Sink/Source: Let $A = \begin{bmatrix} 0 & 1 \\ -c^2 & -2b \end{bmatrix}$, with $b^2 < c^2$. Then

$$u'(t) = v(t)$$
$$v'(t) = -c^2 u(t) - 2bv(t)$$

In this case the eigenvalues of A are $\lambda = -b \pm \sqrt{b^2 - c^2}$, a complex conjugate pair with
negative real parts. Note that

$$c u u'(t) + v v'(t) = cuv - cuv - 2bv^2 = -2bv^2 < 0$$

hence

$$\frac{1}{2} \frac{d}{dt}(c u^2 + v^2) = -bv^2 < 0,$$

and it follows that $R^2(t) = c u(t)^2 + v(t)^2$ is a decreasing function of time. Since $R^2(t)$ is

3

closely related to the square of the distance from the point $(u(t), v(t))$ to the origin, a point on the orbit draws closer to the origin as t increases so this orbit must proceed inwards. Plotting orbits in this case leads to
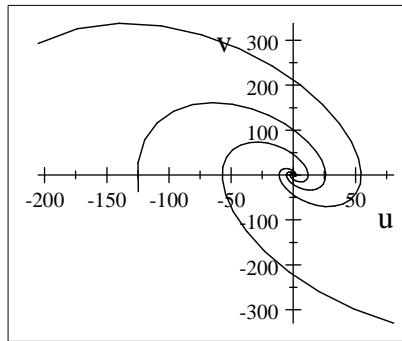

Fig 2 $\operatorname{Re}\lambda < 0$

Since the curve is traced in the clockwise direction, this is a spiral moving in toward the origin and the origin is classified as a spiral sink. In the case that b is negative, but $b^2 < c^2$, the orbit is again a clockwise spiral but in this case, it is moving outward from the origin. This corresponds to the conjugate pair of eigenvalues having a positive real part and in this case the origin is classified as a spiral source. Clearly, the spiral sink is an asymptotically stable critical point while the spiral source is unstable.

3) Saddle Point-   Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

In this case the eigenvalues are $\lambda = \pm 1$ with eigenvectors $\vec{E}_+ = [1,1]^\top$ associated with $\lambda = 1$ and $\vec{E}_- = [-1,1]^\top$ associated with the negative eigenvalue. In particular, since the two real eigenvalues have opposite signs, this causes the orbits to have the following appearance.
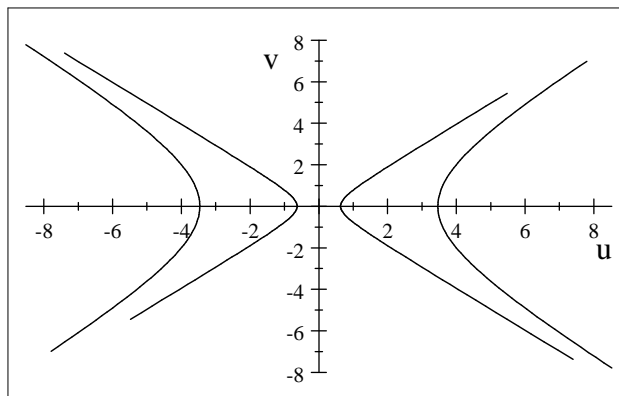

Fig 3 $\lambda_1 < 0 < \lambda_2$

In this picture, the orbits all approach the origin along the direction of the eigenvector $\vec{X}_- = [-1,1]^\top$ associated with the negative eigenvalue. These are the orbits that have negative slope and we refer to this as the direction of approach. The orbits recede from the origin in the direction of the eigenvector $\vec{X}_+ = [1,1]^\top$ associated with the positive eigenvalue. These are the orbits with positive slopes and we refer to this as the escape direction. Since not all trajectories which start near the origin, remain there, a saddle point is an unstable critical point.

4

4) Source/Sink: Let $A = \begin{bmatrix} -1 & 2 \\ 0 & -2 \end{bmatrix}$, eigenvectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \leftrightarrow -1, \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} \leftrightarrow -2$$

In this case the two eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$ are both negative so $u(t)$ and $v(t)$ both tend to zero as t tends to infinity. Then all orbits approach the origin and the trajectories look like
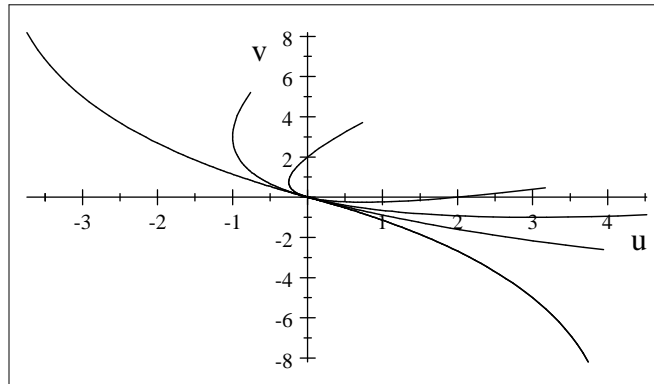

Fig 4 $\lambda_1 < \lambda_2 < 0$

In the case that A has two positive eigenvalues, $u(t)$ and $v(t)$ both grow without bound as t tends to infinity with the result that the trajectories appear as in figure 4 but moving away from the origin. The critical point in this case is called a stable node or sink when the eigenvalues are negative and an unstable node or source if they are positive.

Summarizing, every linear dynamical system has a single critical point located at $(0,0)$ and that critical point must be one of these six types. We are assuming here that A is nonsingular.

4.2.2 Nonlinear Dynamical Systems
A nonlinear dynamical system is a system of the form

$$u'(t) = f(u,v) \qquad (2.2)$$
$$v'(t) = g(u,v)$$

and instead of trying to construct solutions for these systems, we are going to try to sketch families of curves, called orbits or trajectories, $\{(u(t),v(t) : -\infty < t < \infty)\}$, where u and v solve the equations. This is called sketching a phase plane portrait for the system. Here we are assuming that f and g do not depend explicitly on t, and in this case, we say the dynamical system is **autonomous**. Systems which contain explicit dependence on t are called nonautonomous and they are more difficult to analyze. The reason that autonomous systems are simpler is that the orbits are **coherent**. This means that distinct orbits cannot cross each other and the reason they cannot cross is easy to explain. Suppose we have two orbits for the system above that start from distinct points but later cross at a point, (u_*, v_*). If they cross each other then the tangent $\vec{T}_1$ to the first curve does not have the same slope at (u_*, v_*) as $\vec{T}_2$, the tangent to the second curve.
Then if we denote by, $m(T_k)$, the slope of $\vec{T}_k$, $k = 1,2$, at the point (u_*, v_*) where the curves

intersect, then

$$m(T_1) = \frac{dv}{du}]_{C_1} = \frac{v'(t)}{u'(t)}|_{C_1} = \frac{f(u_*, v_*)}{g(u_*, v_*)} = \frac{v'(t)}{u'(t)}|_{C_2} = \frac{dv}{du}]_{C_2} = m(T_2).$$

But this says that since both curves are orbits, their slopes are determined solely by the values of $f(u_*, v_*)$, and $g(u_*, v_*)$. But then the slopes at (u_*, v_*) cannot be different, which is to say, the curves cannot cross.

We have defined a **critical point** for the system to be any point (u_*, v_*), where $f(u_*, v_*) = g(u_*, v_*) = 0$. Then a critical point is a stationary point for the system since $u'(t) = v'(t) = 0$. Now the process of sketching the phase plane portrait for a dynamical system will consist of sketching the local picture for the system in a neighborhood of each of its critical points and then using the fact that the orbits are coherent to blend all the local pictures into a global portrait. We will now describe how we are going to discover the local picture in the neighborhood of a critical point.

Local Analysis
Suppose (u_*, v_*) is a critical point for the dynamical system (2.2) and write the first few terms of the Taylor series expansions for f and g about the point (u_*, v_*),

$$f(u, v) = f(u_*, v_*) + \partial_u f(u_*, v_*)(u - u_*) + \partial_v f(u_*, v_*)(v - v_*) + \cdots$$
$$g(u, v) = g(u_*, v_*) + \partial_u g(u_*, v_*)(u - u_*) + \partial_v g(u_*, v_*)(v - v_*) + \cdots$$

Taking into account that (u_*, v_*) is a critical point, $f(u_*, v_*) = g(u_*, v_*) = 0$, and dropping all the higher order (nonlinear in u and v) terms, we have

$$f(u, v) \approx \partial_u f(u_*, v_*)(u - u_*) + \partial_v f(u_*, v_*)(v - v_*)$$
$$g(u, v) \approx \partial_u g(u_*, v_*)(u - u_*) + \partial_v g(u_*, v_*)(v - v_*)$$

Then the equations of the original dynamical system imply that

$$u'(t) \approx \partial_u f(u_*, v_*)(u - u_*) + \partial_v f(u_*, v_*)(v - v_*)$$
$$v'(t) \approx \partial_u g(u_*, v_*)(u - u_*) + \partial_v g(u_*, v_*)(v - v_*),$$

or, since u_*, v_* are constants,

$$\frac{d}{dt}(u(t) - u_*) \approx \partial_u f(u_*, v_*)(u - u_*) + \partial_v f(u_*, v_*)(v - v_*)$$
$$\frac{d}{dt}(v(t) - v_*) \approx \partial_u g(u_*, v_*)(u - u_*) + \partial_v g(u_*, v_*)(v - v_*).$$

But this is now a linear system of the form

$$\frac{d}{dt}\begin{bmatrix} u(t) - u_* \\ v(t) - v_* \end{bmatrix} = \begin{bmatrix} \partial_u f(u_*, v_*) & \partial_v f(u_*, v_*) \\ \partial_u g(u_*, v_*) & \partial_v g(u_*, v_*) \end{bmatrix} \begin{bmatrix} u(t) - u_* \\ v(t) - v_* \end{bmatrix}$$

where the coefficient matrix

$$J[u, v] = \begin{bmatrix} \partial_u f(u_*, v_*) & \partial_v f(u_*, v_*) \\ \partial_u g(u_*, v_*) & \partial_v g(u_*, v_*) \end{bmatrix}$$

is constant. The only critical point of this linear system is the point (u_*, v_*), where

$$\begin{bmatrix} u(t) - u_* \\ v(t) - v_* \end{bmatrix} = \vec{0}.$$

A famous theorem known as the Hartman-Grobman theorem asserts that the orbit structure for the linear system near its critical point is very close to the orbit structure for the nonlinear system at the point (u_*, v_*). We will illustrate with examples.

Examples
1. Undamped Pendulum
    The dynamical system of this example is a mathematical model for a frictionless pendulum consisting of a mass, m, on a rigid weightless rod of length L. If the angular deflection from the rest position is denoted by θ, then the component of the gravity force in the direction normal to the rod is equal to $-mg \sin\theta$. Then conservation of angular momentum asserts that

$$- mL\theta''(t) = -mg \sin\theta(t).$$

Letting $u(t) = \theta(t)$, $v(t) = \theta'(t)$ leads to $u'(t) = v(t)$, $v'(t) = -\Omega^2 \sin(u(t))$ where $\Omega^2 = g/L$ represents the frequency associated with the oscillating pendulum. For simplicity we will choose $\Omega = 1$ in our example. Then the system, becomes,

$$u'(t) = v(t)$$
$$v'(t) = -\sin(u(t))$$

Then $f(u, v) = v$, $g(u, v) = -\sin(u)$ and the critical points for the system are all points where $v = 0$ and $\sin(u) = 0$; i.e., there are infinitely many critical points, namely all the points $\{(k\pi, 0) : k = any \text{ integer}\}$. Then

$$J[u, v] = \begin{bmatrix} 0 & 1 \\ -\cos(u) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos(k\pi) & 0 \end{bmatrix}$$

We will sketch the local pictures near the critical points $(-\pi, 0), (0, 0)$ and $(\pi, 0)$:

At $(-\pi, 0)$ and $(\pi, 0)$, $\cos(\pm\pi) = -1$, so $J[\pm\pi, 0] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

The eigenvalues in this case are $\lambda = \pm 1$, real and of opposite signs, so the critical point is a saddle point. The eigenvectors are
: $\vec{X}_1 = \{[1, 1]^\top\}$, for $\lambda_1 = 1$, and $\vec{X}_2 = \{[1, -1]^\top\}$ for $\lambda_2 = -1$. Recall that $\vec{X}_1$ is the direction of escape for orbits near this critical point since this is the eigenvector associated with the positive eigenvalue. The other eigenvector is associated with the negative eigenvalue and is therefore the approach direction for this critical point.

At $(0, 0)$, $\cos(0) = 1$, so $J[0, 0] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and in this case the eigenvalues are

$\pm i$. This critical point is a center.

The orbits near a center look like the closed orbits seen in figure 1 while the orbits near a saddle resemble the picture in figure 3. We then try to blend the three local pictures into one global picture using the fact that the orbits are coherent. The result looks like this.
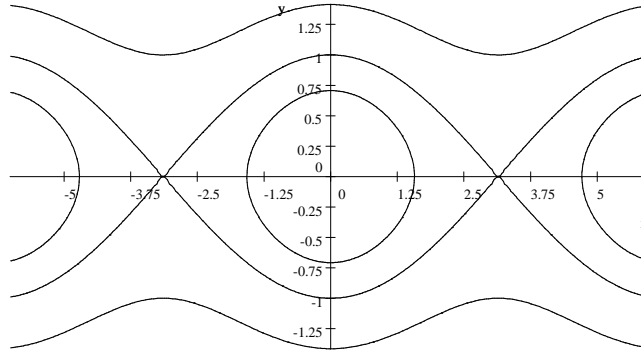

Figure 5 Undamped Pendulum

We can see that there are three kinds of orbits here. There is an orbit in the upper half plane joining the critical point at $(-\pi, 0)$ to the critical point at $(\pi, 0)$ and another orbit in the lower half plane which goes from $(\pi, 0)$ back to $(-\pi, 0)$. These are not two parts of one orbit but are, in fact two different orbits since an orbit that starts near one critical point and goes to another must stop when it reaches the destination critical point (a critical point is a stationary point). However, the curve composed of these two orbits taken together separates the phase plane into two parts and, for this reason, this curve is called a **separatrix** for this system. All the orbits inside the separatrix are closed curves and therefore correspond to periodic solutions of the dynamical system. All the curves lying outside the separatrix are curves that are not closed and in fact assume all values for u as t goes from $-\infty$ to ∞. Along the nonclosed curves in the upper half plane, the variable $v(t)$ varies between two positive values, while along the nonclosed curves in the lower half plane, the variable $v(t)$ varies between two negative values. Notice that the approach and escape directions for the two saddles do, in fact, lie along the directions of the two corresponding eigenvectors and the orbits are traversed in the clockwise direction in conformity with the equations of the system.

Observe that the equations of the system imply that

$$\sin(u)\, u'(t) = \sin(u)\, v$$

$$v\, v'(t) = -\sin(u)\, v$$

hence

$$v\, v'(t) + \sin(u)\, u'(t) = \frac{d}{dt}\left[\frac{1}{2}v(t)^2 + (1 - \cos(u(t)))\right] = 0.$$

Here we have chosen

$$E(u, v) = \frac{1}{2}v(t)^2 + (1 - \cos(u(t)))$$

as the antiderivative for $v\, v'(t) + \sin(u)\, u'(t)$ instead of $G(u, v) = \frac{1}{2}v(t)^2 - \cos(u(t))$, since $E(u, v)$ is zero when $u = v = 0$ and G is not zero. This quantity $E(u, v)$ is an expression which can be interpreted as the total energy in the physical dynamical system and with this choice of E we have calibrated the energy to be zero when the system is "at rest". The equation asserts that the energy is constant along any orbit and along the separtrix in particular, we have $E(\pi, 0) = 2$. Then any orbit for which $E(u, v) < 2$ is a closed orbit corresponding to a

periodic solution and any orbit for which $E(u,v) > 2$ is an orbit outside the separatrix corresponding to an unbounded solution.

Now we can see that the three types of orbits in our example correspond to three different kinds of motions of the pendulum. The closed orbits inside the separatrix correspond to back and forth motions of the pendulum. The small almost circular orbits close to the origin correspond to oscillations of small amplitude and for these orbits, the approximation $\sin(u) \approx u$ is valid. That is, $u''(t) \approx -\Omega^2 u(t)$ on these orbits and this linear equation can be easily solved. The solution is referred to as the small amplitude approximation for the pendulum. Closed orbits of larger amplitude are much less circular and for these orbits the small amplitude approximation is not valid and the solution is not expressible in terms of sines and cosines.

The separatrix corresponds to a motion in which the pendulum starts straight up in the position corresponding to $\theta = \pm\pi$. The pendulum is then given a small push so that it swings down and then back up again so as to once again come to rest in the straight up position. The orbits outside the separatrix correspond to motions in which the pendulum is set in motion with so much energy that the pendulum "loops the loop" and goes over the top. Since there is no mechanism for energy dissipation in the model (i.e., no friction) the pendulum will continue looping forever.

2. The Damped Pendulum

To see what happens when friction is added, modify the momentum equation for the pendulum as follows,

$$- mL\theta''(t) = -mg\sin\theta(t) - c\theta'(t), \qquad c > 0.$$

The term $-c\theta'(t)$ represents a force which is in opposition to the momentum and is proportional to the speed of the pendulum, e.g., a force created by friction with the air. To see that this term does in fact dissipate energy, let $u(t) = \theta(t)$, $v(t) = \theta'(t)$ and write

$$u'(t) = v(t),$$
$$v'(t) = -\Omega^2 \sin(u(t)) - Cv(t), \ C > 0.$$

Then

$$v\,v'(t) + \Omega^2 \sin(u)\,u'(t) = \frac{d}{dt}\left[\frac{1}{2}v(t)^2 + \Omega^2(1 - \cos(u(t)))\right] = -Cv^2 < 0.$$

i.e.,

$$\frac{d}{dt}E(u(t),v(t)) = -Cv^2 < 0,$$

so energy decreases along orbits of this system. To see what the phase plane portrait looks like we note that the new system has the same critical points as the frictionless system. Then (letting $\Omega = 1$, for simplicity) we compute

$$J[u,v] = \begin{bmatrix} 0 & 1 \\ -\cos(u) & -C \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 1 \\ -\cos(k\pi) & -C \end{bmatrix}.$$

At $(-\pi,0)$ and $(\pi,0)$, $\cos(\pm\pi) = -1$, so

$$J[\pm\pi, 0] = \begin{bmatrix} 0 & 1 \\ 1 & -C \end{bmatrix}.$$

Assuming $C^2 < 4$, the eigenvalues are once again real and of opposite signs so this critical point is a saddle point. The eigenvalues can be found to be

$$\lambda_1 = -\frac{1}{2}C - \frac{1}{2}\sqrt{(C^2 + 4)} < 0$$

$$\text{and} \quad \lambda_2 = -\frac{1}{2}C + \frac{1}{2}\sqrt{(C^2 + 4)} > 0.$$

The corresponding eigenvectors are :

$$\vec{X}_1 = \left\{ \begin{bmatrix} \frac{1}{2}C - \frac{1}{2}\sqrt{(C^2 + 4)} \\ 1 \end{bmatrix} \right\}, \; \textit{for } \lambda_1 < 0$$

$$\text{and} \quad \vec{X}_2 = \left\{ \begin{bmatrix} \frac{1}{2}C + \frac{1}{2}\sqrt{(C^2 + 4)} \\ 1 \end{bmatrix} \right\} \; \textit{for } \lambda_2 > 0.$$

At $(0,0)$, $\cos(0) = 1$, so $J[0,0] = \begin{bmatrix} 0 & 1 \\ -1 & -C \end{bmatrix}$, and the eigenvalues are

$$\lambda_1 = -\frac{1}{2}C + \frac{1}{2}\sqrt{(C^2 - 4)},$$

$$\text{and} \quad \lambda_2 = -\frac{1}{2}C - \frac{1}{2}\sqrt{(C^2 - 4)},$$

i.e., the eigenvalues are a complex conjugate pair with negative real part so the critical point is a stable focus with orbits similar to those in figure 2. Then the plot analogous to the frictionless phase plane portrait shows that the closed orbits around the origin have become, in the presence of friction, spirals tending toward the origin (i.e., the rest position). We interpret this to mean that as the pendulum loses energy to friction, the amplitude of the motion decreases to zero. The unbounded orbits in the upper phase plane have become spirals that tend toward the next critical point located at $x = \pi$, while the some of the unbounded orbits in the lower phase plane are attracted to the critical point at the origin.
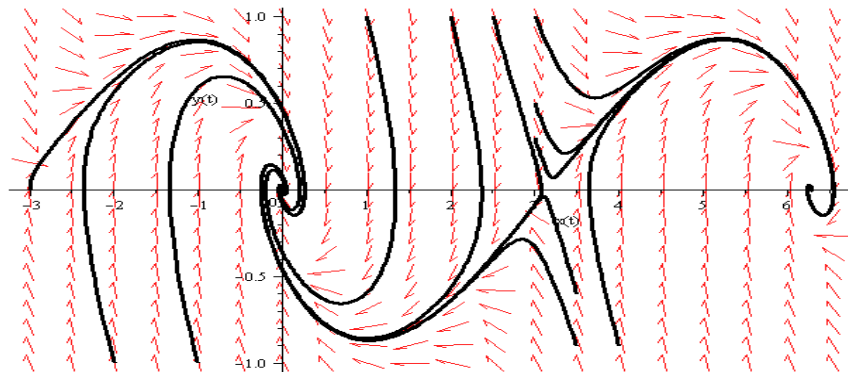

Figure 6 Damped Pendulum

Note that some of the orbits in this picture wind up at the critical point at $(0,0)$, while other orbits eventually go to the critical point at $(2\pi, 0)$. There is also one orbit that comes from

above the horizontal axis and terminates at the saddle point at $(\pi, 0)$. This orbit separates those orbits going to the origin from those orbits which go to the next critical point. The region to the left of this orbit is referred to as the "basin of attraction" for the critical point at $(0,0)$, while the region to the right of this orbit is basin of attraction for the critical point at $(2\pi, 0)$. Of course there are also orbits that are eventually attracted to other critical points so these basins of attraction have some limits to their size.

It is also necessary to note that the orbits shown in Figures 5 and 6 are orbits of the linear approximations to the nonlinear systems. This means that the true orbits may vary slightly from the pictured orbits. Of course a slight perturbation of a spiral is still a spiral, and a slight variation of the vaguely hyperbolic curves near a saddle points are still more or less hyperbolic, but a slight variation of the closed orbits seen in Figure 5 may be no longer closed. For this reason, when a nonlinear system has a critical point which is a center, it is not automatically possible to conclude that the orbits near the center are closed curves even though this is the case for the orbits of the linear approximation to the nonlinear system. Instead, we must find some independent evidence that the orbits are indeed closed curves. In the case of the undamped pendulum, this evidence is provided by the energy expression that asserts that the energy expression, $E(u,v) = \frac{1}{2}v(t)^2 + (1 - \cos(u(t)))$ is constant on orbits of the nonlinear system, and when this constant value is less than 2, this is the equation of a closed curve. In general, it is not always easy to verify that the orbits of a nonlinear system around a center are in fact closed. We say that the source, sink and saddle type of critical points are "structurally stable" while the center is a "structurally unstable" critical point. This just refers to the fact that the orbit pictures around source, sink and saddle points are not significantly altered by slight changes in the system in contrast to the picture near a center where a small change can turn closed orbits into spirals or some other form of non-closed curve. To illustrate this point, consider the system

$$u'(t) = -v(t) + \beta u(t)^3$$
$$v'(t) = u(t)$$

Clearly the only critical point is at $(0,0)$ and $J[u,v] = \begin{bmatrix} 3\beta u^2 & -1 \\ 1 & 0 \end{bmatrix}$. Then the eigenvalue equation for $J(0,0)$ is $\lambda^2 + 1 = 0$, which implies that the critical point is a center. However, $u(t)u'(t) + v(t)v'(t) = \frac{1}{2}\frac{d}{dt}[u^2 + v^2] = \beta u^4$. When β is positive, this implies that a point on an orbit moves away from the origin with increasing t so that the critical point must be a spiral source. When β is negative, the point is a spiral sink. In either case, the origin is a center for the dynamical system but the orbits near the origin are not closed curves.

3. A Predator-Prey Model

We consider two populations, a predator population, whose number at any t is denoted by $P(t)$, and a prey population where $p(t)$ denotes the size of the population at time t. If these populations occupied the same space but did not interact, then the equations governing P and p would be

$$p'(t) = bp(t) - dp(t)$$
$$P'(t) = BP(t) - DP(t)$$

where b, d denote the birth and death rates for $p(t)$ and B and D play the same role for $P(t)$. If the populations interact with the predator population feeding off the prey, then we could

assume d and B depend on P and p respectively. We could assume, for simplicity that $d = kP$, to reflect the fact that the presence of predators increases the death rate of the prey in an amount proportional to the population size of the predators. Similarly, we could suppose the $B = Kp$, to reflect the fact that the presence of plenty of food (in the form of prey) causes the birth rate of the predators to increase proportional to the size of the prey population. Then the system becomes

$$p'(t) = bp(t) - kP(t)p(t)$$
$$P'(t) = KP(t)p(t) - DP(t)$$

For purposes of this example, let us choose simple values for the constants in the system so that we have

$$p'(t) = 6p(t) - 3P(t)p(t)$$
$$P'(t) = 2P(t)p(t) - 4P(t).$$

Then there are critical points at $(0,0)$ and $(2,2)$. We compute the Jacobian matrix,

$$J(x,y) = \begin{bmatrix} 6 - 3P & -3p \\ 2P & 2p - 4 \end{bmatrix}$$

and find

$$J(0,0) = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix},$$

$$J(2,2) = \begin{bmatrix} 0 & -6 \\ 4 & 0 \end{bmatrix}.$$

Then the origin is a saddle point with eigenvectors $\vec{E}(6) = [1,0]^\top$, the escape direction, and $\vec{E}(-4) = [0,1]^\top$, the direction of approach. The point $(2,2)$ is a center and we will show that it is surrounded by closed curve orbits. From the first equation in the system, we see that $p' > 0$ when $P < 2$, and $p' < 0$ when $P > 2$. This tells us that the orbits around the center are traversed in the counter-clockwise direction.

In this example we will be able to show that the orbits around $(2,2)$ are closed curves. We rewrite the system equations in the following separated form

$$\frac{dp}{p(6 - 3P)} = \frac{dP}{P(2p - 4)}$$

$$\text{or} \quad \frac{2p - 4}{p} dp = \frac{6 - 3P}{P} dP$$

The equations are separable and we can integrate to find

$$2p + 3P - 4\ln p - 6\ln P = C.$$

This solution is in implicit form and is not readily recognizable as some well known curve, but it can be plotted and seen to be a closed curve. The implication of this solution is that the two populations will periodically increase and decrease in time, but as long as the closed curve orbit stays away from the origin, neither population will die out. If the value of either population falls below some threshold value required for sustaining reproduction, then both populations die out.

The critical point $\vec{a}$ is called a hyperbolic critical point for the system if $\operatorname{Re}\lambda \neq 0$ for all eigenvalues of $J[\vec{a}]$. Then every hyperbolic critical point is asymptotically stable if it is a sink

and it is unstable if it is a source or a saddle. A center is not a hyperbolic critical point.

Exercises
In each system, find all the critical points and classify them. Sketch the orbits in the neighborhood of each critical point. Try to blend the local portraits into a global phase plane portrait. Identify any periodic solutions or other significant features of the solutions. If a critical point is a center, try to find an implicit solution for closed curve orbits.

1.
$$u'(t) = v(t)$$
$$v'(t) = -u(t) + v(t)^3$$

2.
$$u'(t) = v(t)$$
$$v'(t) = 4u(t) - u(t)v(t)^2$$

3.
$$u'(t) = v(t) - v(t)^2$$
$$v'(t) = u(t)^2 - u(t)$$

4.
$$p'(t) = 6(4 - p(t))p(t) - 3P(t)p(t)$$
$$P'(t) = 2P(t)p(t) - 4P(t).$$

5. A linear dynamical system has a single critical point at the origin. Why? How is that critical point classified?
6. A nonlinear dynamical system may have many (even infinitely many) critical points. What is the purpose of forming the Jacobian matrix and evaluating it at the critical points?
7. How do you determine the directions of escape and approach at a saddle point?
8. At a critical point that is a center are the orbits of the nonlinear system necessarily closed curves? Explain.
9. Which types of critical points are asymptotically stable? Which are stable but not asymptotically stable? Which are unstable?
10. What is a separatrix? What is a basin of attraction?

4.3 Additional Topics
First, let us summarize what we have already found.
● For linear systems, the origin is the only critical point. In 2-dimensions, the origin is a sink if it is a stable node or stable focus and in this case, every orbit of the system is attracted to the origin asymptotically.
● The origin is unstable and repels all orbits if $(0,0)$ is an unstable node or unstable focus or if it is a saddle.
● If the origin is a center, then it is stable (but not asymptotically stable) and all orbits are closed curves around the origin.
● The six possible types of critical point for a linear 2-dimensional system are summarized as follows:

	$\lambda = \lambda_1, \lambda_2$			$\lambda = \alpha \pm i\beta$
stable node	$\lambda_1 < \lambda_2 < 0$ ←sink→	stable focus	$\alpha < 0$
unstable node	$\lambda_1 > \lambda_2 > 0$ ←source→	unstable focus	$\alpha > 0$
saddle	$\lambda_1 > 0 > \lambda_2$	center	$\alpha = 0$

● For nonlinear systems in the plane there can be multiple critical points, each of which can exhibit any of the above behaviors. If the critical points are all found and classified (i.e., identified as one of the 6 types listed above) then the sketch of the phase plane portrait that is consistent with the type of each critical point can be made in the neighborhood of each critical point.

● Finally, using the fact that the orbits are coherent, it should be possible to blend the local pictures into a global phase plane portrait for the nonlinear system.

Limit Cycles

Although an approach based on these few points is successful on many examples, there are other examples where this is not sufficient. The approach fails for example when the system has a critical point that is encircled by a simple closed curve called a limit cycle. To illustrate this situation, consider the system

$$x'(t) = y + x(1 - x^2 - y^2)$$
$$y'(t) = -x + y(1 - x^2 - y^2).$$

The origin is the only critical point for this system and it turns out to be an unstable focus, a source. By multiplying the first equation by x and the second by y and adding, we find
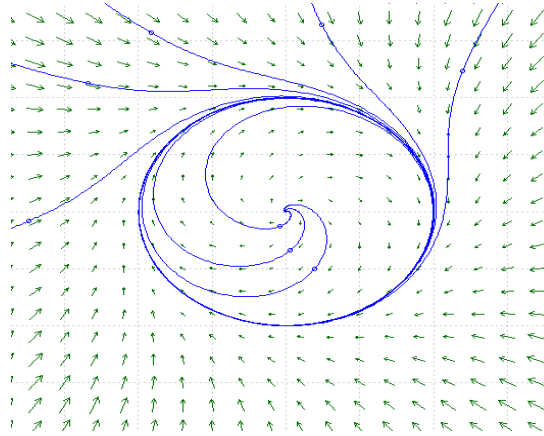
$$x(t)x'(t) + y(t)y'(t) = (x^2 + y^2)(1 - x^2 - y^2),$$

or

$$\frac{1}{2}\frac{d}{dt}(x^2 + y^2) = (x^2 + y^2)(1 - x^2 - y^2).$$

Now $r^2 = x^2 + y^2$ is the square of the distance from the origin to $(x(t), y(t))$, a point on an orbit and expressing the last equation in terms of r leads to

$$\frac{1}{2}\frac{d}{dt}r^2(t) = r^2(1 - r^2).$$

Then $r(t)$ is increasing as long as $r < 1$ and it is decreasing when $r > 1$. If we let Ω denote the disc $\{(x, y) : x^2 + y^2 \leq 1\}$, then any orbit that originates in Ω can never leave Ω (since in order to cross the circle $r = 1$ we would have to have $\frac{d}{dt}r^2(t) > 0$ at $r = 1$ which is impossible because of the equation). Similarly, if we let Ω denote the annular region $\{(x, y) : 1 < x^2 + y^2 \leq 4\}$, then any orbit that originates in Ω must remain there ($r(t)$ is steadily decreasing in Ω and yet no orbit can cross the inner boundary of Ω). Then orbits in this annular region spiral inward toward a limit cycle. The phase plane picture is the following.

Limit Cycle

In this example we can actually solve explicitly for $r(t)$. If we let $u(t) = r(t)^2$, then $u'(t) = 2u(1 - u)$ and if we separate and integrate we find

$$\ln u - \ln(1 - u) = 2t + C_0$$
$$\text{or} \qquad \frac{u}{1 - u} = C_1 e^{2t}$$

and

$$u(t) = r(t)^2 = \frac{C_1}{e^{-2t} + C_1} \to 1 \text{ as } t \to \infty \text{ for any } C_1.$$
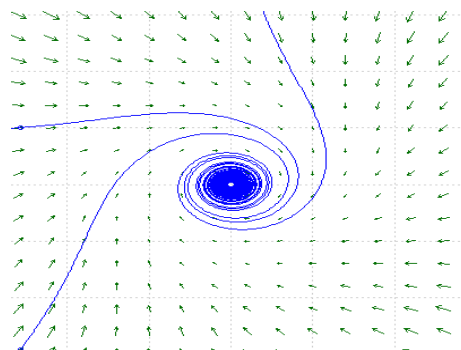
This shows that $r(t)$ tends to 1 for all orbits (the initial point of the orbit determines C_1 but here the value of the constant has no effect on the limit of $r(t)$ as t increases).

If we modify the example to read

$$x'(t) = y + x(-x^2 - y^2)$$
$$y'(t) = -x + y(-x^2 - y^2).$$

then the critical point at the origin becomes a center but the orbits of the nonlinear system are not closed curves about the critical point. The origin is a spiral sink for this example with the following phase plane picture,

Spiral Sink

Since in this example we can show that

$$\frac{1}{2} \frac{d}{dt} r^2(t) = -r(t)^4,$$

this behavior is not unexpected.

A less contrived example of a limit cycle arises from the so called Van der Pol equation,

$$x''(t) - \left(1 - x(t)^2\right)x'(t) + x(t) = 0$$

which translates into the following first order dynamical system,

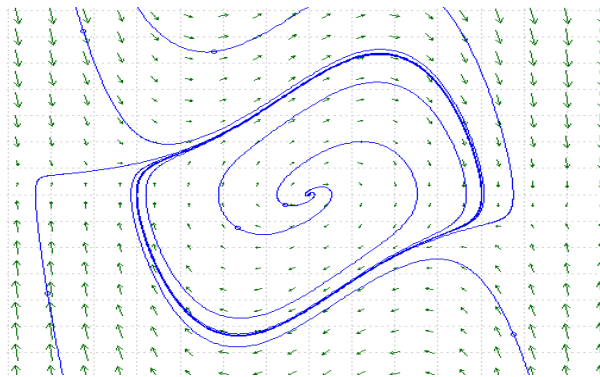$$u'(t) = v(t)$$
$$v'(t) = -u(t) + \left(1 - u(t)^2\right)v(t)$$

This system has a single critical point, a spiral source at the origin. However, the equations of the system imply that

$$u(t)u'(t) + v(t)v'(t) = \left(1 - u(t)^2\right)v(t)^2$$

or

$$\frac{1}{2}\frac{d}{dt}(u^2 + v^2) = (1 - u^2)v^2.$$

It follows from this last equation that $u^2 + v^2$ is increasing when $u^2 < 1$ but increasing for $u^2 > 1$,. suggesting the presence of a limit cycle. The phase plane picture associated with this system is as follows,

which clearly shows the presence of a limit cycle.

Equations that Separate

In general, as the previous examples illustrate, it is often difficult to determine what the phase plane portrait looks like around a center type critical point for a nonlinear dynamical system. There are special cases, however, when the situation becomes clear. Consider the following example:

$$\dot{x}(t) = 4 - 2y(t)$$
$$\dot{y}(t) = 12 - 3x(t)^2$$

This system has critical points at $(-2,2)$ and $(2,2)$. We find

$$J(x,y) = \begin{bmatrix} 0 & -2 \\ -6x & 0 \end{bmatrix}$$

and then:

$A(2,2)$ has eigenvalues $\lambda_+ = \sqrt{24}$, and $\lambda_- = -\sqrt{24}$ $(2,2)$ is a saddle

$A(-2,2)$ has eigenvalues $\lambda_1 = i\sqrt{24}$, and $\lambda_2 = -i\sqrt{24}$ $(-2,2)$ is a center

We are now faced with the problem of deciding whether or not there are closed orbits around the center at $(-2,2)$. If we write

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{12 - 3x^2}{4 - 2y}$$

then $\int [4 - 2y]dy = \int [12 - 3x^2]dx;$

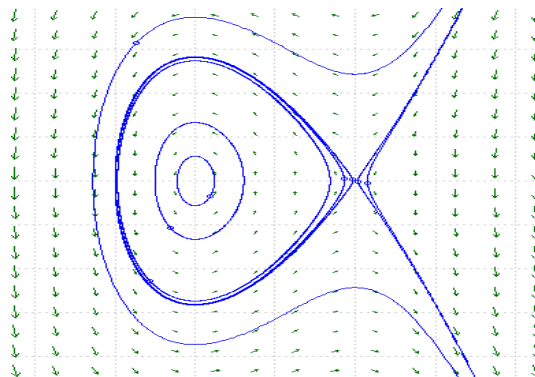Integration leads to $4y - y^2 = 12x - x^3 + C$, or $(y - 2)^2 + x(12 - x^2) = C$. This means that for

$$F(x,y) = (y - 2)^2 + x(12 - x^2),$$

we have

$$\frac{d}{dt}F(x(t), y(t)) = \partial_x F \cdot \dot{x}(t) + \partial_y F \cdot \dot{y}(t)$$

$$= (12 - 3x^2)(4 - 2y) + (2y - 4)(12 - 3x^2)$$

$$= 0$$

Then F is constant on orbits of the dynamical system which is the same as saying the level curves of $F(x,y)$ are the orbits of the dynamical system. For example, $F(2,2) = 16$, hence the curve $(y - 2)^2 + x(12 - x^2) = 16$ is the orbit that passes through the saddle at $(2,2)$. Plotting this and several other level curves looks like this

Since the first equation of the system implies that, $\dot{x} < 0$ when $y > 2$, we can see that the closed orbits around $(-2,2)$ must be traversed in the counter clockwise direction. The orbit, C, passing through the saddle point begins and ends at the same critical point. This orbit is called a homoclinic orbit and is furthermore a **separatrix** for the system, since it separates the closed orbits (inside C) from the orbits that are not closed curves (these are outside C). The region inside of C is called a **trapping region** for the system since any system orbit that begins inside this region never leaves the region. The closed curve orbits in the trapping region correspond to periodic solutions for the dynamical system since the state variables $(x(t), y(t))$ continue to pass through the same set of points over and over as t goes from $-\infty$ to $+\infty$. This example shows that when the system is sufficiently simple to allow us to solve, even implicitly, for the orbits of the system, then the question of how the orbits look around a center can be settled by simply plotting the orbits (in this example, the level curves of F). In general, this simple approach will not be possible.

Other Simple Systems

There are several important physical systems whose mathematical models lead to dynamical systems of the following general form:

$$\dot{x} = x(p - ax - by)$$
$$\dot{y} = y(q - cx - dy)$$

Here $a, b, c, d,$ p and q denote constant parameters. Models leading to systems of this form include:

1. **Predator-prey models**- Here the predator population, $y(t)$, survives by eating the prey, whose population, $x(t)$, has a maximum sustainable level of X_∞. The term αxy represents the rate at which the prey population is diminished by the predators (the prey decrease rapidly when there are a lot of predators present). The term βxy reflects the fact that the predators increase more rapidly when there is a sufficient food supply present in the form of prey.

$$\dot{x} = k\left(1 - \frac{x}{X_\infty}\right)x - \alpha xy$$
$$\dot{y} = (\beta x - d)y$$

2. **Competition models**- Here there are two populations competing for the same, finite, pool of resources. Each population has a maximum sustainable level and the terms αxy, βxy represent the interaction between the two populations as they compete for the limited available resources.

$$\dot{x} = r\left(1 - \frac{x}{X_\infty}\right)x - \alpha xy$$
$$\dot{y} = s\left(1 - \frac{y}{Y_\infty}\right)y - \beta xy$$

3. **Combat models**- Here the variables x and y represent troop levels for two armies. The linear terms ax and by represent losses due to noncombat causes (e.g. accidents, illness) while the terms αxy, βxy represent losses due to encounters with the enemy.

$$\dot{x} = -ax - \alpha xy$$
$$\dot{y} = -by - \beta xy$$

In order to analyze models of the general form

$$\dot{x} = x(p - ax - by)$$
$$\dot{y} = y(q - cx - dy)$$

We make use of **null clines**; i.e., curves along which $\dot{x} = 0$ (x null clines) or on which $\dot{y} = 0$ (y null clines). For our system,

$$\dot{x} = 0 \quad if \quad x = 0 \ \ or \ \ p - ax - by = 0$$
$$\dot{y} = 0 \quad if \quad y = 0 \ \ or \ \ q - cx - dy = 0$$

Apparently, there are four null clines; the $y -$ axis and the line, $ax + by = p$ are x null clines,

while the y null clines are the x- axis and the line, $cx + dy = q$. At a point where two null clines meet, we have $\dot{x} = \dot{y} = 0$, so intersection points of x null clines with y null clines determine the critical points for the system. For our example we have critical points at:

$$1) \quad x = 0 \qquad\qquad y = 0$$
$$2) \quad x = 0 \qquad\qquad y = q/d$$
$$3) \quad y = 0 \qquad\qquad x = p/a$$
$$4) \quad x = \frac{pd - bq}{ad - bc} \qquad y = \frac{qa - cp}{ad - bc} \qquad if \quad \det = ad - bc \neq 0$$

The critical point 4) at the point where the two lines meet will turn out to be either a saddle point or a stable node. If it is a saddle, the critical points 2) and 3) then turn out to be stable nodes. In this case, all orbits originating in the first quadrant tend to one of the critical points 2) or 3) where one of the populations is extinct. If the critical point 4) is a stable node, then all orbits originating in the first quadrant tend to this critical point, and the populations stabilize at this cooperative equilibrium point where neither population is extinct.

We consider two examples of this type:

$$\dot{x} = x(1 - x - y)$$
$$\dot{y} = y(3 - 2x - 4y)$$

Here, we have critical points at:

$$1) \quad x = 0 \qquad\qquad y = 0$$
$$2) \quad x = 0 \qquad\qquad y = 3/4$$
$$3) \quad y = 0 \qquad\qquad x = 1$$
$$4) \quad x = 1/2 \qquad\quad y = 1/2$$
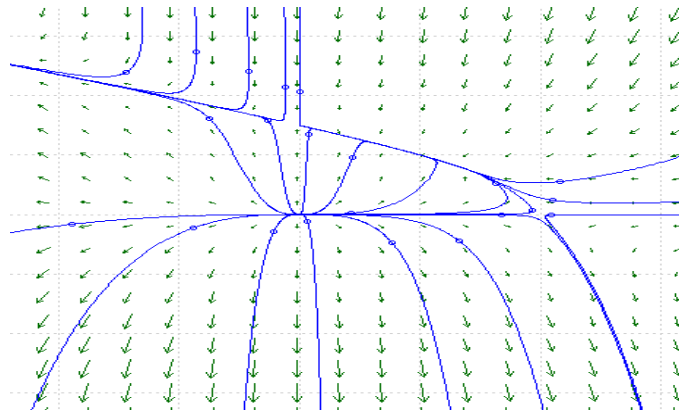
We find

$$J(x, y) = \begin{bmatrix} 1 - 2x - y & -x \\ -2y & 3 - 2x - 8y \end{bmatrix}$$

and

$$J(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \text{ has eigenvalues } \lambda = 1, 3 \qquad \text{Unstable Node}$$

$$J(0, 3/4) = \begin{bmatrix} 1/4 & 0 \\ -3/2 & -3 \end{bmatrix} \text{ has eigenvalues } \lambda = -3, 1/4 \qquad \text{Saddle}$$

$$J(1, 0) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \text{ has eigenvalues } \lambda = -1, 1 \qquad \text{Saddle}$$

$$J\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{bmatrix} -\tfrac{1}{2} & -\tfrac{1}{2} \\ -1 & -2 \end{bmatrix} \text{ has eigenvalues } \lambda = \frac{-5 \pm \sqrt{17}}{4} \qquad \text{Stable Node}$$

The phase plane picture looks like this,

from which it is evident that every orbit originating in the first quadrant is attracted to the critical point at $(.5, .5)$. On the other hand consider the example

$$\dot{x} = x(1 - x - y)$$
$$\dot{y} = y(2 - 3x - y)$$

In this case, we have critical points at:

$$\begin{array}{lll} 1) & x = 0 & y = 0 \\ 2) & x = 0 & y = 2 \\ 3) & y = 0 & x = 1 \\ 4) & x = 1/2 & y = 1/2 \end{array}$$

Here,

$$J(x, y) = \begin{bmatrix} 1 - 2x - y & -x \\ -3y & 2 - 3x - 2y \end{bmatrix}$$

and
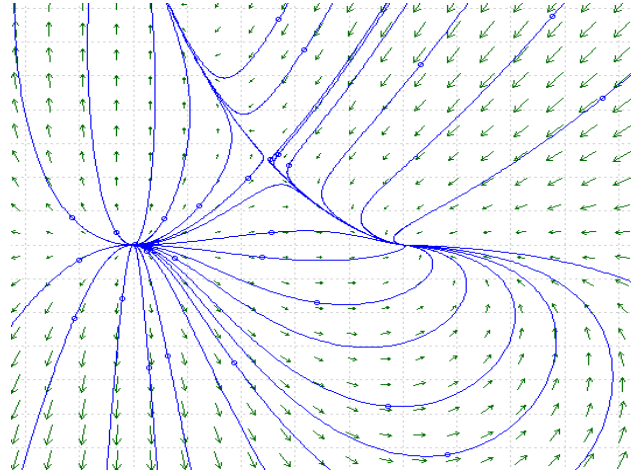
$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$ has eigenvalues $\lambda = 1, 2$ Unstable Node

$$J(0,2) = \begin{bmatrix} -1 & 0 \\ -6 & -2 \end{bmatrix}$$ has eigenvalues $\lambda = -1, -2$ Stable Node

$$J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$ has eigenvalues $\lambda = -1, -1$ Stable Node

$$J\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{bmatrix} -\tfrac{1}{2} & -\tfrac{1}{2} \\ -\tfrac{3}{2} & -\tfrac{1}{2} \end{bmatrix}$$, has eigenvalues $\lambda = -\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}, \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}$ Saddle

Now the phase plane picture looks like this,

In this case all orbits that originate in the first quadrant are attracted to one of the two stable nodes $(0,2)$ or $(1,0)$ where one or the other of the populations is extinct. This means that this system has no cooperative equilibrium point where the two populations can coexist. Note that in the first example the determinant $ad - bc$ for the system of equations to find the point of intersection of the two lines had the value $+2$ while in the second example $ad - bc = -2$. It is possible to show in general that when the determinant is positive, the intersection point of the lines is a stable node and when the determinant is negative, the intersection point is a saddle.

Liapunov functions

Another tool for examining the stability of nonhyperbolic critical points is the so called **Liapunov function**. For example, the previously considered energy function, $E(u, v)$, for the nonlinear pendulum is constant along orbits for the undamped pendulum while $E(u, v)$ decreases along orbits for the damped pendulum. In the damped case the orbits cross the level curves for energy (i.e., the curves along which energy is constant) in the direction of decreasing energy. The energy function for the nonlinear pendulum is a special case of a Liapunov function. Liapunov functions can be used not only in the case of 2-dimensional systems but can be extended to higher dimensions as well. We will present the discussion in the two dimensional setting but we will also give some examples in three dimensions.

Suppose that the system

$$x'(t) = F(x(t), y(t))$$
$$y'(t) = G(x(t), y(t))$$

has a nonhyperbolic critical point at $\vec{a} = (x_0, y_0)$. Suppose also that we have a function $E = E(x, y)$ which is a smooth function on R^2 such that

$$E(x_0, y_0) = 0, \qquad \text{and} \qquad E(x, y) > 0 \quad if \quad (x, y) \neq (x_0, y_0).$$

Then

i) If $\frac{d}{dt}E(x(t), y(t)) < 0$ *on all orbits of the system*
   then $\vec{a} = (x_0, y_0)$ *is* asymptotically stable

ii) If $\frac{d}{dt}E(x(t),y(t)) \leq 0$ *on all orbits of the system*
      then $\vec{a} = (x_0,y_0)$ *is* stable
iii) If $\frac{d}{dt}E(x(t),y(t)) > 0$ *on all orbits of the system*
      then $\vec{a} = (x_0,y_0)$ *is* unstable

Of course the success of this method depends on being able to find a function E, which is called a Liapunov function for the system. There are no universal recipes for finding Liapunov functions but by studying the examples, we can at least get some idea of how to find an E. Note that on orbits $(x(t),y(t))$ of the system,

$$\tfrac{d}{dt}E(x(t),y(t)) = \partial E/\partial x \; x'(t) + \partial E/\partial y \; y'(t) = \; F \, \partial_x E + G \partial_y E$$

and if the expression on the right is zero then the derivative is zero. This means that E is constant along all orbits of the system; i.e., orbits of the system are level curves of the function E . If this derivative is not zero but instead is negative for all orbits of the system, this means that the orbits are not level curves of E but as a point moving along an orbit cuts a level curve, it goes from the exterior of the level curve into the interior. Finally, if this derivative is positive for all orbits of the system, the orbits are not level curves of E but as a point moving along an orbit cuts a level curve, it goes from the interior of the level curve into the exterior.

   To illustrate, consider the system

$$x'(t) = -y^3$$
$$y'(t) = x^3$$

The only critical point for this system is $(0,0)$ and the Jacobian has only the eigenvalue zero at this point. Then this is a nonhyperbolic critical point. Now consider the function $E(x,y) = x^4 + y^4$. For this E we have

$$E(0,0) = 0 \quad and \quad E(x,y) > 0 \quad for \; (x,y) \neq (0,0)$$

and

$$\tfrac{d}{dt}E(x(t),y(t)) = 4x^3x'(t) + 4y^3y'(t) = 0.$$

This means that level curves of E are orbits of the system and the origin is stable but not asymptotically stable.
   Now consider the 3-dimensional example

$$x'(t) = -2y + yz$$
$$y'(t) = x - xz$$
$$z'(t) = xy.$$

Then $(0,0,0)$ is the only critical point and we have

$$J(0,0,0) = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ with eigenvalues: } \lambda = 0, i\sqrt{2}, -i\sqrt{2}$$

Then the origin is a nonhyperbolic critical point. If we try an E of the form

$$E(x,y,z) = ax^2 + by^2 + cz^2,$$

then

$$\tfrac{d}{dt}E(x(t),y(t)) = 2ax(-2y + yz) + 2by(x - xz) + 2cz(xy)$$

$$= (2b - 4a)xy + 2(a - b + c)xyz$$

and if we choose the constants such that $b = 2a$, $c = a > 0$ then
$E(0,0,0) = 0$, $E > 0$ *for* $(x,y,z) \neq (0,0,0)$, *and* $\tfrac{d}{dt}E(x(t),y(t)) = 0$ on all orbits of the system;
i.e., all orbits are level curves for the ellipsoid $x^2 + 2y^2 + z^2 = C^2$. This is a stable critical
point but it is not asymptotically stable.

If we change the system slightly to

$$x'(t) = -2y + yz - x^3$$
$$y'(t) = x - xz - y^3$$
$$z'(t) = xy - z^3$$

then the critical point is the same and J has the same eigenvalues at the critical point. The
same Liapunov function, $E = x^2 + 2y^2 + z^2$, now satisfies
$E(0,0,0) = 0$, $E > 0$ *for* $(x,y,z) \neq (0,0,0)$, *and*

$$\tfrac{d}{dt}E(x(t),y(t)) = -2(x^4 + 2y^4 + z^4) < 0 \quad \textit{for } (x,y,z) \neq (0,0,0).$$

Then the origin is an asymptotically stable critical point for this system even though all of the
eigenvalues have zero real part.

Finally consider the second order equation

$$x''(t) + q(x) = 0,$$

where $q(x)$ denotes a smooth function such that $q(0) = 0$ *and* $xq(x) > 0$ *for* $x \neq 0$. For
example, the pendulum equation with $q(x) = \sin(x)$ satisfies this for $-\pi < x < \pi$. If we let
$x'(t) = y(t)$ then

$$x'(t) = y(t)$$
$$y'(t) = -q(x)$$

and there is a critical point at the origin. If we let

$$E(x,y) = \tfrac{1}{2}y^2 + \int_0^x q(s)\,ds$$

then
$$\tfrac{d}{dt}E(x(t),y(t)) = yy' + q(x)x' = -yq(x) + q(x)y = 0.$$

Then the orbits of the system are level curves of E (which is just the total energy function) and the origin is therefore a stable critical point.